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arXiv:hep-th/0701181v1 19 Jan 2007

Estimative for the size of the compactification radius of a one extra dimension Universe

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January 22, 2007

Abstract

In this work, we use the Casimir effect to probe the existence of one extra dimension. We begin by evaluating the Casimir pressure between two plates in a $M^4 \times S^1$ manifold, and then use an appropriate statistical analysis in order to compare the theoretical expression with a recent experimental data and set bounds for the compactification radius.

1 Introduction

It is fair to say that, in a broad sense, the search for unification is the greatest enterprize of theoretical physics. It started a long time ago, when Sir Isaac Newton showed that celestial and terrestrial mechanics could be described by the same laws, and reached one of its highest peaks in the second half of the nineteenth century, when electricity, magnetism and optics were all gathered into Maxwell equations.

The quest for unification continued, and, in a historical paper at 1919 [1], T. Kaluza managed to combine classical electromagnetism and gravitation into a single, very elegant scheme. The downside was that his theory required an extra spacial dimension, for which there was no evidence whatsoever. Some years later, O. Klein pushed the idea a little further [2], proposing, among other things, a circular topology of a very tiny radius for the extra dimension, maybe at the Planck scale region. Although it presented a great unification appeal, the Kaluza-Klein idea has been left aside for several decades. Only in the mid-seventies, due to the birth of supergravity theory [3], the extra dimensions came back to the theoretical physics scenario. As supergravity also had its own problems, it seemed that the subject would be washed out again, but, less than a decade later, the advent of string and superstring theories [4] made it a cornerstone in extremely high energy physics. Nowadays, with the development of M-theory [5] and some associated ideas, like the cosmology of branes [6], it might even be said that extra dimensions are almost a commonplace in modern physics.

As the Casimir effect [7] has a strong dependence with the space-time dimensionality, it has been suggested [8] that Casimir force experiments may be a powerful tool to detect the existence of extra dimensions. In a recent paper, Poppenhaeger et al. [9] carried out a calculation in order to set bounds for the size of an hypothetic extra dimension, but relied their work upon the data of the classical experiment of M. Sparnaay [10], which, compared to modern measurements, is of very crude precision. Thus, inspired by the previous discussion, the goal of this work is to estimate the size of one extra dimension. We begin by evaluating the Casimir pressure between two plates in a hypothetical universe with a $M^4 \times S^1$ topology. We use the standard mode summation formula for the Casimir effect, and the calculations are carried out within the analytical regularization scheme, which is closely related to some generalized zeta functions. The result for the Casimir energy and pressure show an explicit dependence on the distance between the plates and on the S^1 radius, as they should. As our final task, we use some recent experimental data [11] in order to set limits for the values of the compactification radius.

2 The Casimir effect in a $M^4 \times S^1$ spacetime

Let us begin by writing the line element of the $M^4 \times S^1$ universe

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - r^2\, d\theta^2, \tag{1}$$

where r is the S^1 radius. Due to the simplicity of this metric, the field equations in this manifold are essentially the same as the minkovskian ones. This holds in particular for the massless vectorial field, and so we have

$$\partial_\mu F^{\mu\nu} = 0, \qquad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \tag{2}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3}$$

In the radiation gauge we may write

$$A_0 = 0, \quad \partial_\mu A^\mu = 0, \tag{4}$$

and so the field equation may be recast into

$$\Box A^\mu = 0 \tag{5}$$

Let us assume that the conducting plates are at the planes $x = 0$ and $x = a$. This setup leads to the following boundary conditions (BC)

$$F^{\mu\nu}|_{x=0} = F^{\mu\nu}|_{x=a} = 0 \quad \text{if} \;\; \mu \neq 1,\, \nu \neq 1. \tag{6}$$

The S^1 topology also imposes a periodicity condition for the electromagnetic field

$$A^\mu(x^4) = A^\mu(x^4 + 2\pi r). \tag{7}$$

Now we have to solve equation (5) constrained by conditions (6) e (7). That is a straightforward task, so we merely quote the eigenmodes

$$A_1 \;\; = \;\; A_1^{(0)} \cos\left(\frac{m_1 \pi x}{a}\right) \mathrm{e}^{i(\vec{k}_\perp \cdot \vec{x}_\perp + n\theta - \omega t)}, \;\; m_1 = 0, 1, 2, ...$$

$$
\begin{aligned}
A_j &= i\,A_j^{(0)}\mathrm{sen}\left(\frac{m_j\pi x}{a}\right)\mathrm{e}^{i(\vec{k}_\perp\cdot\vec{x}_\perp+n\theta-\omega t)},\ \ m_j=1,2,...\\
\omega^2 &= \left(\frac{m\pi}{a}\right)^2+\left(\frac{n}{r}\right)^2+k_\perp^2,\quad j=2,3,4;\quad n=0,\pm1,\pm2... \qquad (8)
\end{aligned}
$$

The gauge condition (4) leads us to

$$
A_1^{(0)}\frac{m\pi}{a}+\mathbf{A^{(0)}}\cdot\mathbf{k}_\perp+\frac{\mathbf{n}}{\mathbf{r}}\mathbf{A_4^{(0)}}=\mathbf{0} \qquad (9)
$$

In a arbitrary manifold, the Casimir energy of a given quantum field is, in general, not given by a simple mode-summation formula, even if it is a free field [1]. Fortunately for us, it may be shown that for the $M^4\times S^1$ universe the vacuum energy is indeed given by the usual sum of modes

$$
\begin{aligned}
\mathcal{E}(a,r) &= \frac{\hbar}{2}\sum_{\mathbf{k}\lambda}\omega_{\mathbf{k}\lambda}=\frac{\hbar c L^2}{8\pi^2}\int d^2\mathbf{k}_\parallel\sum_{n=-\infty}^{\infty}\left[\sqrt{\left(\frac{n}{r}\right)^2+k_\parallel^2}\right.\\
&+ \left.p\sum_{m=1}^{\infty}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n}{r}\right)^2+k_\parallel^2}\right], \qquad (10)
\end{aligned}
$$

where p is the number of possible polarizations of the photon ($p=3$ in this case). The previous expression is purely formal, since its r.h.s. is infinite. So, in order to proceed, we introduce a cut-off parameter s in (10). Then

$$
\begin{aligned}
\mathcal{E}_{reg}(a,r;s) &= \frac{L^2\hbar c}{4\pi}\int_0^\infty k_\parallel dk_\parallel\sum_{m=-\infty}^{\infty}\left\{p\sum_{n=1}^{\infty}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n}{r}\right)^2+k_\parallel^2\right]^{\frac{1-s}{2}}\right.\\
&+ \left.\left[\left(\frac{n}{r}\right)^2+k_\parallel^2\right]^{\frac{1-s}{2}}\right\}. \qquad (11)
\end{aligned}
$$

Performing the integral in $k_\parallel$ we arrive at

$$
\begin{aligned}
\mathcal{E}_{reg}(a,r;s) &= \frac{\hbar c L^2 p}{4\pi(s-3)}\left(\frac{a}{\pi}\right)^{s-3}\left[\sum_{m=1}^{\infty}m^{3-s}+2\sum_{n,m=1}^{\infty}\left(m^2+\left(\frac{na}{\pi r}\right)^2\right)^{\frac{3-s}{2}}\right]\\
&+ \frac{\hbar c L^2}{2\pi(s-3)}r^{s-3}\sum_{n=1}^{\infty}n^{3-s}. \qquad (12)
\end{aligned}
$$

Let us now recall the definition of the Epstein functions [13], and, as a particular case, the Riemann zeta function[14]

$$
E_N(s;a_1,...,a_N)=\sum_{n_1,...,n_N=1}^{\infty}\left[a_1n_1^2+...+a_Nn_N^2\right]^{-s},\ \ \zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\,. \qquad (13)
$$

By using these definitions, we may recast expression (12) into

$$
\mathcal{E}_{reg}(a,r;s) = \frac{\hbar c L^2 p}{4\pi(s-3)}\left(\frac{a}{\pi}\right)^{s-3}\left[\zeta(s-3)+2\,E_2\left(\frac{s-3}{2};1,\frac{a^2}{\pi^2r^2}\right)\right]
$$

[1] For interacting fields, the mode-summation fails to produce the Casimir energy even in flat Minkowski space-time [12]. As a matter of fact, even for free fields in a Minkowski universe it's not always obvious that the vacuum energy is given by a sum of modes [19].

$$+ \quad \frac{\hbar c L^2 r^{s-3}}{2\pi(s-3)}\,\zeta(s-3)\,. \tag{14}$$

The Epstein functions have a well known analytical continuation, which were thoroughly studied in [13, 14], among other references. As a more detailed discussion of that matter would take us too far afield, let us merely quote the analytic continuation of the Epstein function $E_2(s; a_1, a_2)$

$$\begin{aligned} E_2\left(s; a_1^2, a_2^2\right) = -\frac{a_1^{-2s}}{2}\zeta(2s-1) + \frac{\sqrt{\pi}}{2a_2}\frac{\Gamma(s-1/2)}{\Gamma(s)}a_1^{1-2s}\zeta(2s-1) \\ + \quad \frac{2\pi^s}{\Gamma(s)}\sum_{n_1,n_2=1}^{\infty} a_2^{-s-1/2}\left(\frac{n_1}{a_1 n_2}\right)^{s-1/2} K_{s-1/2}\left(\frac{2\pi a_1 n_1 n_2}{a_2}\right)\,, \end{aligned} \tag{15}$$

where $K_\nu(x)$ stands for the modified Bessel function. The reflection formula for the Riemann zeta function will also be very useful

$$\zeta(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1-s)\,. \tag{16}$$

It is now a straightforward matter put (14) into the form

$$\begin{aligned} \mathcal{E}_{reg}(a,r;s) \quad &= \quad p\frac{\hbar c L^2}{4\pi(s-3)}\frac{1}{\Gamma\left(\frac{s-3}{2}\right)}\left[\frac{a^{s-3}}{\sqrt{\pi}}\Gamma\left(2-\frac{s}{2}\right)\zeta(4-s)\right. \\ &+ \quad \frac{a r^{s-4}}{\pi^{5-s}}\Gamma\left(\frac{5-s}{2}\right)\zeta(5-s) \\ &+ \quad \left.\frac{4a^{\frac{s}{2}-1}}{\sqrt{\pi}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mr}{n}\right)^{\frac{s}{2}-2}K_{\frac{s}{2}-2}\left(\frac{2mna}{r}\right)\right] \\ &- \quad (p-2)\frac{\hbar c L^2}{4(s-3)}\frac{\Gamma\left(2-\frac{s}{2}\right)}{\Gamma\left(\frac{s-3}{2}\right)}\frac{r^{s-3}}{\pi^{\frac{9}{2}-s}}\zeta(4-s)\,, \end{aligned} \tag{17}$$

and, in the limit of $s \to 0$, we get

$$\begin{aligned} \mathcal{E}(a,r) \quad &= \quad -p\frac{\hbar c L^2 \pi^2}{1440 a^3} + (p-2)\frac{\hbar c L^2}{1440\pi r^3} - 2p\pi r L^2\frac{3\hbar c}{128\pi^7}\frac{a}{r^5}\zeta(5) \\ &- \quad p\frac{\hbar c L^2}{4\pi^2 r^2 a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{n}{m}\right)^2 K_2\left(\frac{2mna}{r}\right) \end{aligned} \tag{18}$$

Due to renormalization issues, we now have to evaluate the Casimir energy of the region defined by the plates, but with no plates whatsoever. This calculation is analogous to the one leading to (18), so we merely state the result

$$\mathcal{E}_{ED}(a,r) = -2p\pi r L^2\frac{3\hbar c}{128\pi^7}\frac{a}{r^5}\zeta(5)\,. \tag{19}$$

Then, subtracting this term from (18), we finally obtain the Casimir energy

$$\mathcal{E}_{Cas}(a,r) = \quad - \quad p\frac{\hbar c L^2 \pi^2}{1440 a^3} - (p+2)\frac{\hbar c L^2}{1440\pi r^3}$$

$$- \quad p\frac{\hbar c L^2}{4\pi^2 r^2 a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{n}{m}\right)^2 K_2\left(\frac{2mna}{r}\right). \tag{20}$$

This is an important result by itself, but, if we want to make some comparison with the experiments, we need an expression for the Casimir pressure. Fortunately, the relation between the Casimir energy and pressure is a simple one

$$\mathcal{P}(a,r) = -\frac{1}{L^2}\frac{\partial \mathcal{E}_{Cas}}{\partial a} = -p\frac{\pi^2\hbar c}{480a^4} - p\frac{\hbar c}{4\pi^2 r^2 a^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[3\left(\frac{n}{m}\right)^2 K_2\left(\frac{2mna}{r}\right) + 2\frac{n^3 a}{mr}K_1\left(\frac{2mna}{r}\right)\right], \tag{21}$$

where we used some recurrence relations among the modified Bessel functions [16]. If we now make $p = 2$ in expressions (20) and (21) and take the limiting case of $r \to 0$, we will get respectively the standard Casimir energy and pressure obtained in [7].

3 Estimative of the compactification radius

The plane geometry is by far the simplest to work with in theoretical calculations. Unfortunately, the situation is not so friendly from the experimental point of view. As a matter of fact, it is quite difficult to obtain a satisfactory precision (for modern standards) by using parallel plates. For this reason, we feel that it is important to mention some peculiarities of the experiment that we will rely on.

First of all, it is a recent experiment that uses two parallel plates in order to detect the Casimir force [11]. The most popular setup nowadays for measuring the Casimir effect is the sphere-above-a-plate configuration [12, 15], due to its great precision rate. A notable distinction between these two setups is the optimized distance for measuring the force: approximately 0.5 to 1 μm with the plate-plate configuration, against the usual range of 0.1 to 0.5 μm with the sphere-plate setup.

The apparatus itself used in that experiment is also very interesting. The two parallel 'plates' are simulated by the opposing faces two silicon beams. One of these beams is rigidly connected to a frame, in a such a way to provide an accurate control of the distance between the two beams. The other beam is a thin cantilever that plays the part of a resonator, since it is free to oscillate around its clamping point. The apparatus is designed to measure the square plates oscillating frequency shift ($\Delta\nu^2$), that is related to the Casimir pressure in the following way [18]

$$\Delta\nu^2 = \nu^2 - \nu_0^2 = -\frac{L^2}{4\pi^2 m_{eff}}\frac{\partial \mathcal{P}}{\partial a}, \tag{22}$$

where m_{eff} is the effective mass of the resonator.

Substituting (21) in the previous expression, we get

$$\Delta\nu^2(a,r) \quad = \quad -p\frac{\hbar c L^2}{4\pi^2 m_{eff}}\left\{\frac{\pi^2}{120a^5}\right.$$

$$
+ \quad \frac{1}{\pi^2 a r} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\left(3 \frac{n}{m^3 a^3} + \frac{5}{2} \frac{n^3}{m a r^2} \right) K_1 \left(\frac{2mna}{r} \right) \right.
$$
$$
+ \quad \left. \left. \left(3 \frac{n^2}{m^2 r a^2} + \frac{n^4}{r^3} \right) K_0 \left(\frac{2mna}{r} \right) \right] \right\} . \tag{23}
$$

Now that we have a theoretical expression of $\Delta\nu^2$ as a function of a and r, we will fit r using the least square method and the experimental data of [11]. As we are fitting just one parameter, we can estimate the best value for r from the graph on figure 1 just by looking for the value of r that leads to a minimum value of χ^2.

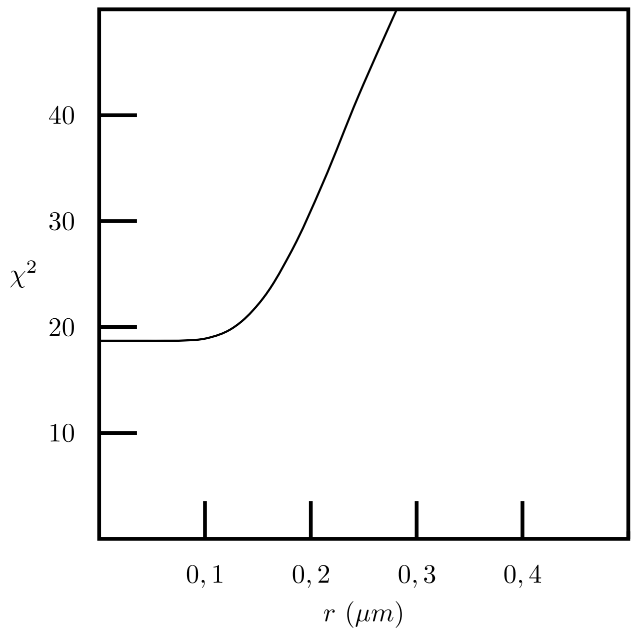

Figure 1: Graph of χ^2 *versus* r.

Our fit for the compactification radius produced the value of 0^{+123}_{-0} nm, and the uncertainties on r give the upper and lower bounds for this radius. In a successful fit, the minimum value of χ^2 should coincide, approximately, with the number of degrees of freedom of the fit. As in this case we have 8 degrees of freedom[2], and the minimum for χ^2 came out to be 18.6, we can state that no good agreement was obtained between the theoretical model and the experimental data.

4 Conclusion

In this article, we have used the Casimir effect to probe the existence of one extra dimension. We started by evaluating the Casimir pressure between two

[2]The degree of freedom of a fit is defined as being the subtraction of the number of experimental points used in the fit by the number of adjusted variables. In this case, we have 9 experimental points and one adjusted variable, which gives the degree of freedom aforementioned.

perfect conducting plates living in a 4+1 universe, given in (21), where the extra dimension is compactified in a S^1 topology. In order to set bounds for the compactification radius, we proceeded to the comparison of this result with the experimental data of [11], and, after an appropriate statistical analysis, this procedure showed that the best value for the compactification radius is between 0 and 123nm.

We know that the results for the Minkowski space-time are in close agreement with the experimental data [12]. In order to be consistent with this picture, the extra compactified dimension should contribute as a small perturbation to the four-dimensional result, but, as we have seen, this is not the case. Among other things, the extra dimension led to a new polarization degree for the electromagnetic field, which essentially bumped the M^4 result by a factor of approx. 3/2, that is not small. It is important to say that this new polarization freedom does not allow the $r \to 0$ limit to be taken carelessly, for it represents the transition from $M^4 \otimes S^1$ to M^4, in which a polarization degree is discontinuously lost.

We finish by saying that there are other corrections to the Casimir effect, such as the finite conductivity and some thermal effects [20], which for sure are more important than the existence of an extra dimension. Besides that, there is the roughness of the plate material [21] and possibly some edge effects [22], which, if necessary, should also be considered. Hence, in a more rigorous approach, these influences should be taken into account, and the comparison should be made with more accurate experiments [23].

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